JOURNAL OF
PURE AND
APPLIED ALGEBRA

ELSEVIER Journal of Pure and Applied Algebra 133 (1998) 143–149

The composition of dualities in a nondegenerate Morita context

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Abstract

The relations between the lattices of submodules of members of an Morita context are studied. The pairs of reversing order maps are defined, which determine the dualities between the sets of 'closed' submodules. In the rather weak conditions these dualities can be composed obtaining the projectivities defined by simple maps. If the context is nondegenerate then the square is shown consisting of four dualities and two projectivities co-ordinated between them. © 1998 Elsevier Science B.V. All rights reserved.

1991 Math. Subj. Class.: 16D90

0. Introduction

Let $(R, {}_RM_S, {}_SN_R, S)$ be an arbitrary Morita context with the bimodule morphisms

$$(\,,\,): M \otimes_S N \to R, \qquad [\,,\,]: N \otimes_R M \to S.$$

We study the relations between the lattices of submodules of members of this context. In [5, 3] the preserving order maps and respective lattice isomorphisms were shown. Hutchinson [1] indicated a projectivity and a duality in the strong conditions (for example $(M,N) = R$). The similar results were obtained in [2]. The essential weakening of necessary conditions with the help of torsion theories is indicated in [6, 4].

In the present work we investigate the reversing-order maps between the lattices of submodules $\mathbb{L}({}_RM)$, $\mathbb{L}(N_R)$ and $\mathbb{L}(S_S)$. They have the form of annihilators and are defined by module structure of M and N, as well as by morphisms $(,)$ and $[,]$. In a natural way appear the Galois connections and accompanying dualities (reversing-order bijections). For two dualities with the common part the composition can be formed, which determine a projectivity (preserving-order bijection). To obtain such situation

PII: S0022-4049(97)00190-4

we need only some comparatively modest conditions: the 'faithfulness' of M and N both in ordinary sense and relative to the operations $(,)$ and $[,]$. If the context is nondegenerate these conditions are satisfied and we obtain a square with two diagonals consisting of harmonically acting four dualities and two projectivities (Theorem 13).

For any module ${}_RX$ we denote by $\mathbb{L}({}_RX)$ the lattice of all submodules of ${}_RX$. For an arbitrary Morita context $(R, {}_RM_S, {}_SN_R, S)$ with bimodule morphisms $(,): M \otimes_S N \to R$ and $[,]: N \otimes_R M \to S$ we consider the following pairs of maps between the lattices of submodules:

$$
\begin{tikzcd}[row sep=huge, column sep=huge]
\mathbb{L}({}_RM) \arrow[r, shift left, "r'"] \arrow[d, bend right=40, "\gamma_M"'] \arrow[d, shift right, "\alpha_M"'] \arrow[dr, shift left, "G_M"] & \mathbb{L}(N_R) \arrow[l, shift left, "l'"] \arrow[d, bend right=40, "\gamma_N"'] \arrow[d, shift right, "\alpha_N"'] \arrow[dl, shift right, "G_N"'] \\
\mathbb{L}({}_SS) \arrow[r, shift left, "r"] \arrow[u, shift right, "\beta_M"'] \arrow[u, bend right=40, "\delta_M"'] \arrow[ur, shift right, "Q_N"'] & \mathbb{L}(S_S) \arrow[l, shift left, "l"] \arrow[u, shift right, "\beta_N"'] \arrow[u, bend right=40, "\delta_N"'] \arrow[ul, shift left, "Q_M"]
\end{tikzcd}
\tag{1}
$$

where

$$\alpha_M({}_RK) = \{s \in S \mid Ms \subseteq K\}, \qquad \beta_M({}_SJ) = \{m \in M \mid [N,m] \subseteq J\},$$
$$\gamma_M({}_RK) = [N,K], \qquad \delta_M({}_SJ) = MJ,$$
$$G_M({}_RK) = \{s \in S \mid Ks = 0\}, \qquad Q_M({}_SJ) = \{m \in M \mid mJ = 0\},$$
$$r({}_SJ) = \{s \in S \mid Js = 0\}, \qquad l(J_S) = \{s \in S \mid sJ = 0\},$$
$$r'({}_RK) = \{n \in N \mid (K,n) = 0\}, \qquad l'(L_R) = \{m \in M \mid (m,L) = 0\}.$$

The maps with indexes N are defined similarly

$$\alpha_N(L_R) = \{s \in S \mid sN \subseteq L\}, \qquad \beta_N(J_S) = \{n \in N \mid [n,M] \subseteq J\},$$
$$\gamma_N(L_R) = [L,M], \qquad \delta_N(J_S) = JN,$$
$$G_N(L_R) = \{s \in S \mid sL = 0\}, \qquad Q_N({}_SJ) = \{n \in N \mid Jn = 0\}.$$

It is obvious that the vertical maps are order preserving, the rest of maps being order reversing. We will study separately the four triangles of diagram (1).

1. Triangle formed by the pairs $(\alpha_M, \beta_M), (G_M, Q_M)$ and (r, l)

By definitions we have the relations:

$${}_RK \subseteq Q_M G_M({}_RK), \quad J_S \subseteq G_M Q_M(J_S), \quad {}_SJ \subseteq lr({}_SJ), \quad J_S \subseteq rl(J_S),$$

therefore the pairs (r, l) and (G_M, Q_M) form the Galois connections. If we denote

$$\operatorname{Im} G_M = \{G_M({}_RK) \mid K \in \mathbb{L}({}_RM)\},$$

then

$$\operatorname{Im} G_M = \{J \in \mathbb{L}(S_S) \mid J = G_M Q_M(J)\}$$

and similarly

$$\operatorname{Im} Q_M = \{K \in \mathbb{L}({}_R M) \mid K = Q_M G_M(K)\}.$$

So we obtain the following dualities between the sets of 'closed' submodules:

$$\mathbb{L}({}_S S) \supseteq \operatorname{Im} l \underset{l}{\overset{r}{\rightleftarrows}} \operatorname{Im} r \subseteq \mathbb{L}(S_S), \tag{2}$$

$$\mathbb{L}({}_R M) \supseteq \operatorname{Im} Q_M \underset{Q_M}{\overset{G_M}{\rightleftarrows}} \operatorname{Im} G_M \subseteq \mathbb{L}(S_S). \tag{3}$$

To compose these dualities in $\mathbb{L}(S_S)$ we need the relation $\operatorname{Im} r = \operatorname{Im} G_M$, which can be obtained in the case when the maps r and G_M are expressed by one another. With this aim we consider the auxiliary maps γ_M and δ_M and the following conditions of faithfulness of modules M and N:

(A) $Ms = 0$ implies $s = 0$ (i.e. M_S is faithful),

(B) $[N, M] = 0$ implies $m = 0$.

Lemma 1. *Condition* (A) *implies the equality* $r = G_M \delta_M$ *and condition* (B) *implies the equality* $G_M = r\gamma_M$. *Therefore, conditions* (A) *and* (B) *ensure the equality* $\operatorname{Im} r = \operatorname{Im} G_M$ *and so dualities* (2) *and* (3) *can be composed obtaining the projectivity*

$$\mathbb{L}({}_R M) \supseteq \operatorname{Im} Q_M \underset{Q_M r}{\overset{lG_M}{\rightleftarrows}} \operatorname{Im} l \subseteq \mathbb{L}({}_S S). \tag{4}$$

Proof. By definitions

$$G_M \delta_M({}_S J) = \{s \in S \mid MJs = 0\}$$

and

$$r({}_S J) = \{s \in S \mid Js = 0\},$$

so $r({}_S J) \subseteq G_M \delta_M({}_S J)$ and condition (A) yields the equality. Similarly,

$$r\gamma_M(K) = \{s \in S \mid [N, K]s = [N, Ks] = 0\}$$

and

$$G_M(K) = \{s \in S \mid Ks = 0\},$$

therefore, condition (B) implies the equality $r\gamma_M(K) = G_M(K)$. If both conditions (A) and (B) are satisfied then $r = G_M \delta_M$ implies $\operatorname{Im} r \subseteq \operatorname{Im} G$ and $G_M = r\gamma_M$ implies $\operatorname{Im} G_M \subseteq \operatorname{Im} r$. So we have $\operatorname{Im} r = \operatorname{Im} G_M$ what permits to compose dualities (2) and (3) obtaining projectivity (4). $\square$

In the same conditions (A) and (B) the maps of projectivity (4) are of simple form: they coincide with the maps α_M and β_M restricted to $\operatorname{Im} Q_M$ and $\operatorname{Im} l$, respectively.

Lemma 2. *Condition* (A) *implies the equality* $l = \alpha_M Q_M$, *therefore*, $\alpha_M(K) = lG_M(K)$ *for every* $K \in \operatorname{Im} Q_M$. *Condition* (B) *implies the equality* $Q_M = \beta_M l$, *therefore* $\beta_M(J) = Q_M r(J)$ *for every* $J \in \operatorname{Im} l$.

Proof. From definitions

$$\alpha_M Q_M(J) = \{s \in S \mid MsJ = 0\},$$

$$l(J) = \{s \in S \mid sJ = 0\}$$

and by condition (A)

$$\alpha_M Q_M(J) = l(J),$$

i.e. $l = \alpha_M Q_M$. If $K \in \operatorname{Im} Q$ then $K = Q_M G_M(K)$ and

$$\alpha_M(K) = \alpha_M Q_M G_M(K) = lG_M(K).$$

The second affirmation is proved similarly. □

From Lemmas 1 and 2 immediately follows:

Proposition 3. *If conditions* (A) *and* (B) *are satisfied then dualities* (2) *and* (3) *can be composed and the resulting projectivity* (4) *is determined by the maps* α_M *and* β_M:

$$\begin{tikzcd}
\operatorname{Im} Q_M \arrow[r, shift left, "G_M"] \arrow[rrr, bend left, "\alpha_M"] & \operatorname{Im} G_M \arrow[l, shift left, "Q_M"] \arrow[r, phantom, "="] & \operatorname{Im} r \arrow[r, shift left, "l"] & \operatorname{Im} l. \arrow[l, shift left, "r"] \arrow[lll, bend left, "\beta_M"]
\end{tikzcd}$$

2. The triangle formed by the pairs (α_N, β_N), (G_M, Q_M) and (r', l')

The maps r' and l' possess the properties ${}_RK \subseteq l'r'({}_RK)$ and $L_R \subseteq r'l'(L_R)$, so they determine the Galois connection and the associated duality:

$$\mathbb{L}({}_RM) \supseteq \operatorname{Im} l' \underset{l'}{\overset{r'}{\rightleftarrows}} \operatorname{Im} r' \subseteq \mathbb{L}(N_R). \tag{5}$$

To compose it in $\mathbb{L}({}_RM)$ with duality (3) the relation $\operatorname{Im} l' = \operatorname{Im} Q_M$ is necessary, which can be obtained in the following conditions:

(A′) $rM = 0$ implies $r = 0$ (i.e. ${}_RM$ is faithful);
(B′) $(m, N) = 0$ implies $m = 0$.

Lemma 4. *Condition* (A′) *implies the equality* $l' = Q_M \gamma_M$ *and condition* (B′) *implies the equality* $Q_M = l' \delta_N$. *Therefore, conditions* (A′) *and* (B′) *yield the relation* $\operatorname{Im} l' = \operatorname{Im} Q_M$ *which permits to compose dualities* (3) *and* (5) *obtaining the projectivity*

$$\mathbb{L}(N_R) \supseteq \operatorname{Im} r' \underset{r'Q_M}{\overset{G_M l'}{\rightleftarrows}} \operatorname{Im} G_M \subseteq \mathbb{L}(S_S). \tag{6}$$

As in the previous case we can simplify these maps by conditions (A′) and (B′), obtaining $G_M l' = \alpha_N$ and $r' Q_M = \beta_N$ on $\operatorname{Im} r'$ and $\operatorname{Im} G_M$, respectively.

Lemma 5. *Condition* (A′) *implies the equality* $r' = \beta_N G_M$, *therefore,*

$$\beta_N(J_S) = r' Q_M(J_S)$$

for every $J \in \operatorname{Im} G_M$. *Condition* (B′) *implies the equality* $G_M = \alpha_N r'$, *therefore,*

$$\alpha_N(L_R) = G_M l'(L_R)$$

for every $L \in \operatorname{Im} r'$.

From Lemmas 4 and 5 we have the following conclusion:

Proposition 6. *If conditions* (A′) *and* (B′) *are satisfied then dualities* (3) *and* (5) *can be composed and the resulting projectivity* (6) *is determined by the maps* α_N *and* β_N:

$$\begin{tikzcd} \operatorname{Im} r' \arrow[r, shift left, "l'"] \arrow[rr, bend left=40, "\alpha_N"] & \operatorname{Im} l' = \operatorname{Im} Q_M \arrow[l, shift left, "r'"] \arrow[r, shift left, "G_M"] & \operatorname{Im} G_M. \arrow[l, shift left, "Q_M"] \arrow[ll, bend left=40, "\beta_N"] \end{tikzcd}$$

3. The triangle formed by the pairs (α_N, β_N), (G_N, Q_N) and (r, l)

The pair of maps (G_N, Q_N) determine the Galois connection and the duality

$$\mathbb{L}(S_S) \supseteq \operatorname{Im} G_N \underset{G_N}{\overset{Q_N}{\rightleftarrows}} \operatorname{Im} Q_N \subseteq \mathbb{L}(N_R) \tag{7}$$

which is compatible with duality (2) in $\mathbb{L}({}_SS)$ if $\operatorname{Im} G_N = \operatorname{Im} l$. To obtain such relation we consider the conditions:
(C) $sN = 0$ implies $s = 0$ (i.e. ${}_SN$ is faithful),
(D) $[n, M] = 0$ implies $n = 0$.

Lemma 7. *Condition* (C) *implies the equality* $l = G_N\delta_N$ *and condition* (D) *implies the equality* $G_N = l\gamma_M$. *Therefore, conditions* (C) *and* (D) *yield the relation* $\operatorname{Im} G_N = \operatorname{Im} l$, *which permits to compose dualities* (2) *and* (7) *obtaining the projectivity*

$$\mathbb{L}(N_R) \supseteq \operatorname{Im} Q_N \underset{Q_N l}{\overset{rG_N}{\rightleftarrows}} \operatorname{Im} r \subseteq \mathbb{L}(S_S). \tag{8}$$

In conditions (C) and (D) the maps of this projectivity coincide with α_N and β_N restricted to $\operatorname{Im} Q_M$ and $\operatorname{Im} r$, respectively.

Lemma 8. *Condition* (C) *implies the equality* $r = \alpha_N Q_N$, *therefore*, $\alpha_N(L_R) = rG_N(L_R)$ *for every* $L \in \operatorname{Im} Q_N$. *Condition* (D) *implies the equality* $Q_N = \beta_N r$, *therefore*, $\beta_N(J_S) = Q_N l(J_S)$ *for every* $J \in \operatorname{Im} r$.

Proposition 9. *If conditions* (C) *and* (D) *are satisfied then dualities* (2) *and* (7) *can be composed and the resulting projectivity* (8) *is determined by the maps* α_N *and* β_N:

$$\begin{tikzcd} \operatorname{Im} Q_M \arrow[r, shift left, "G_N"] \arrow[rrr, bend left, "\alpha_N"] & \operatorname{Im} G_N \arrow[l, shift left, "Q_N"] \arrow[r, phantom, "="] & \operatorname{Im} l \arrow[r, shift left, "r"] & \operatorname{Im} r. \arrow[l, shift left, "l"] \arrow[lll, bend left, "\beta_N"] \end{tikzcd}$$

4. The triangle formed by the pairs (α_M, β_M), (G_N, Q_N) and (r', l')

In this case we have dualities (5) and (7), and to compose them in $\mathbb{L}(N_R)$ we need the relation $\operatorname{Im} r' = \operatorname{Im} Q_N$. We consider conditions:

(C′) $Nr = 0$ implies $r = 0$ (i.e. N_R is faithful),

(D′) $(M, n) = 0$ implies $n = 0$.

Lemma 10. *Condition* (C′) *implies the equality* $r' = Q_N\gamma_M$ *and condition* (D′) *implies the equality* $Q_N = r'\delta_M$. *Therefore the conditions* (C′) *and* (D′) *ensure the relation* $\operatorname{Im} r' = \operatorname{Im} Q_N$ *which permits to compose dualities* (5) *and* (7) *obtaining the projectivity*

$$\mathbb{L}({}_RM) \supseteq \operatorname{Im} l' \underset{l'Q_N}{\overset{G_N r'}{\rightleftarrows}} \operatorname{Im} G_N \subseteq \mathbb{L}({}_SS). \tag{9}$$

Lemma 11. *Condition* (C′) *implies the equality* $l' = \beta_M G_N$, *therefore* $\beta_M({}_SJ) = l'Q_N$ *for every* $J \in \operatorname{Im} G_N$. *Condition* (D′) *implies the equality* $G_N = \alpha_M l'$, *therefore* $\alpha_M({}_RK) = G_N r'({}_RK)$ *for every* $K \in \operatorname{Im} l'$.

Proposition 12. *If conditions* (C′) *and* (D′) *are satisfied then dualities* (5) *and* (7) *can be composed and the resulting projectivity* (9) *is determined by the maps* α_M *and* β_M:

$$\begin{tikzcd}
\mathrm{Im}\, l' \arrow[r, shift left, "r'"] \arrow[rr, bend left, "\alpha_M"] & \mathrm{Im}\, r' = \mathrm{Im}\, Q_M \arrow[l, shift left, "l'"] \arrow[r, shift left, "G_N"] & \mathrm{Im}\, G_N. \arrow[l, shift left, "Q_M"] \arrow[ll, bend left, "\beta_M"]
\end{tikzcd}$$

Combining Propositions 3, 6, 9 and 12, we deduce the following result:

Theorem 13. *If the context* $(R, {}_RM_S, {}_SN_R, S)$ *is nondegenerate (i.e. conditions* (A), (B), (A′), (B′), (C), (D), (C′) *and* (D′) *are satisfied) then we have the following situation*:

$$\begin{tikzcd}
\mathrm{Im}\, l' = \mathrm{Im}\, Q_M \arrow[rr, shift left, "r'"] \arrow[d, shift right, "\alpha_M"'] \arrow[drr, shift left, "G_M"] & & \mathrm{Im}\, Q_N = \mathrm{Im}\, r' \arrow[ll, shift left, "l'"] \arrow[d, shift right, "\alpha_N"'] \arrow[dll, shift left, "Q_N"] \\
\mathrm{Im}\, l = \mathrm{Im}\, Q_N \arrow[rr, shift left, "r"] \arrow[u, shift right, "\beta_M"'] \arrow[urr, shift left, "G_N"] & & \mathrm{Im}\, G_N = \mathrm{Im}\, r, \arrow[ll, shift left, "l"] \arrow[u, shift right, "\beta_N"'] \arrow[ull, shift left, "Q_M"]
\end{tikzcd}$$

where all maps are bijections: four compatible dualities (r, l), (G_M, Q_M), (r', l'), (G_N, Q_N), *and two projectivities* (α_M, β_M), (α_N, β_N) *which are the compositions of respective dualities.*

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